

TOPOLOGY OF THE MILNOR FIBRATIONS OF POLAR WEIGHTED HOMOGENEOUS POLYNOMIALS

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ABSTRACT. Let P be a 2-variable polar weighted homogeneous polynomial and let F_t be a deformation of P which is also a polar weighted homogeneous polynomial. If $|F_t|$ is a Morse function on the orbit space of the S^1 -action, then the handle decomposition obtained by this Morse function induces a round handle decomposition of the Milnor fibration of F_t . In the present paper, we describe a round handle decomposition of the Milnor fibration of F_t concretely and give the number of round handles by the number of positive and negative components of the links of singularities appearing before and after the deformation. We also give a formula of characteristic polynomials of these singularities by using the decomposition of the monodromy of the Milnor fibration induced by a round handle decomposition.

1. INTRODUCTION

We consider a polynomial of complex variables $\mathbf{z} = (z_1, \dots, z_n)$ which is given by

$$P(\mathbf{z}, \bar{\mathbf{z}}) := \sum_{i=1}^m c_i \mathbf{z}^{\nu_i} \bar{\mathbf{z}}^{\mu_i},$$

where $c_i \in \mathbb{C}^*$, $\mathbf{z}^{\nu_i} = z_1^{\nu_{i,1}} \cdots z_n^{\nu_{i,n}}$ and $\bar{\mathbf{z}}^{\mu_i} = \bar{z}_1^{\mu_{i,1}} \cdots \bar{z}_n^{\mu_{i,n}}$ for $\nu_i = (\nu_{i,1}, \dots, \nu_{i,n})$ and $\mu_i = (\mu_{i,1}, \dots, \mu_{i,n})$ respectively. Here $\bar{\mathbf{z}}^{\mu_i}$ represents the complex conjugate of $\mathbf{z}^{\mu_i} = z_1^{\mu_{i,1}} \cdots z_n^{\mu_{i,n}}$. We call $P(\mathbf{z}, \bar{\mathbf{z}})$ a *mixed polynomial* of complex variables $\mathbf{z} = (z_1, \dots, z_n)$. A point $\mathbf{w} \in \mathbb{C}^n$ is called a *mixed singular point* of $P(\mathbf{z}, \bar{\mathbf{z}})$ if the gradient vectors of $\Re P$ and $\Im P$ are linearly dependent at \mathbf{w} [14, 15]. Suppose that $P(0, \dots, 0) = 0$ and P has an isolated singularity at the origin. There exist positive real numbers ε and δ with $\delta \ll \varepsilon \ll 1$ such that the map

$$P : D_\varepsilon^{2n} \cap P^{-1}(\partial D_\delta^2) \rightarrow \partial D_\delta^2$$

is a locally trivial fibration over ∂D_δ^2 , where $D_\varepsilon^{2n} = \{\mathbf{z} \in \mathbb{C}^n \mid \|\mathbf{z}\| \leq \varepsilon\}$ and $D_\delta^2 = \{\eta \in \mathbb{C} \mid |\eta| \leq \delta\}$. This map is called *the Milnor fibration of P at the origin*.

Ruas, Seade and Verjovsky [16] and Cisneros-Molina [2] introduced the following classes of mixed polynomials. Let p_1, \dots, p_n and q_1, \dots, q_n be integers such that $\gcd(p_1, \dots, p_n) = \gcd(q_1, \dots, q_n) = 1$. We define the S^1 -action and the \mathbb{R}^* -action on \mathbb{C}^n as follows:

$$s \circ \mathbf{z} = (s^{p_1} z_1, \dots, s^{p_n} z_n), \quad r \circ \mathbf{z} = (r^{q_1} z_1, \dots, r^{q_n} z_n),$$

where $s \in S^1$ and $r \in \mathbb{R}^*$. A mixed polynomial $P(\mathbf{z}, \bar{\mathbf{z}})$ is called a *polar weighted homogeneous polynomial* if there exists a positive integer d_p such that $P(\mathbf{z}, \bar{\mathbf{z}})$ satisfies

$$P(s^{p_1} z_1, \dots, s^{p_n} z_n, \bar{s}^{p_1} \bar{z}_1, \dots, \bar{s}^{p_1} \bar{z}_n) = s^{d_p} P(\mathbf{z}, \bar{\mathbf{z}}), \quad s \in S^1.$$

The weight vector (p_1, \dots, p_n) is called *the polar weight* and d_p is called *the polar degree*. Similarly $P(\mathbf{z}, \bar{\mathbf{z}})$ is called a *radial weighted homogeneous polynomial* if there exists a positive integer d_r such that

$$P(r^{q_1} z_1, \dots, r^{q_n} z_n, r^{q_1} \bar{z}_1, \dots, r^{q_n} \bar{z}_n) = r^{d_r} P(\mathbf{z}, \bar{\mathbf{z}}), \quad r \in \mathbb{R}^*.$$

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The integer d_r is called *the radial degree*. If P is polar and radial weighted homogeneous, P admits the global Milnor fibration $P : \mathbb{C}^n \setminus P^{-1}(0) \rightarrow \mathbb{C}^*$ and the monodromy of the Milnor fibration is given by the S^1 -action, see for instance [16, 2, 14, 15].

We study the topology of the Milnor fibration of a mixed polynomial P by using a deformation of P . Here a *deformation of P* is a polynomial map $F : \mathbb{C}^n \times \mathbb{R} \rightarrow \mathbb{C}, (\mathbf{z}, t) \mapsto F_t(\mathbf{z})$, with $F_0(\mathbf{z}) = P(\mathbf{z}, \bar{\mathbf{z}})$. A deformation of P is useful to study the Milnor fibration of P . For complex isolated singularities, it is known that there exist a neighborhood U of the origin of \mathbb{C}^n and a deformation F_t of a complex polynomial $P(\mathbf{z})$ such that $F_t(\mathbf{z})$ is also a complex polynomial and any singularity of $F_t(\mathbf{z})$ in U is a Morse singularity for any $0 < t \ll 1$, see for instance [9, Chapter 4]. A sufficiently small compact neighborhood of each Morse singularity can be regarded as a $2n$ -dimensional n -handle and thus we have a decomposition

$$D_\varepsilon^{2n} \cap F_t^{-1}(D_\delta^2) \cong (D_\varepsilon^{2n} \cap F_t^{-1}(D_{\delta_t}^2)) \cup_\varphi (\sqcup_{i=1}^\ell (n\text{-handle})_i),$$

where ℓ is the Milnor number of the singularity of P at $(0, \dots, 0)$, $\varphi = (\varphi_1, \dots, \varphi_\ell)$ is the ℓ -tuple of the attaching map φ_i of $(n\text{-handle})_i$ and $D_{\delta_t}^2$ is a 2-disk centered at 0 with radius δ_t such that $\delta_t < \delta$ and $F_t|_{F_t^{-1}(D_{\delta_t}^2)}$ has no singularities. Note that the framing of each handle attaching is determined by the vanishing cycle of the corresponding Morse singularity [7]. Such a decomposition induces a decomposition of the monodromy of the Milnor fibration into those of the Morse singularities.

In this paper, we observe analogous deformations for mixed singularities. Let P be a 2-variable polar and radial weighted homogeneous polynomial which has an isolated singularity at the origin of \mathbb{C}^2 and let F_t be a deformation of P . Set $S_k(F_t) = \{\mathbf{z} \in U \mid \text{rank } dF_t(\mathbf{z}) = 2 - k\}$ for $k = 1, 2$. We assume that F_t satisfies the following properties:

- (1) F_t is polar weighted homogeneous for any $0 \leq t \ll 1$, which implies that, for each $0 < t \ll 1$, the singular set $S_1(F_t) \cup S_2(F_t)$ consists of the union of a finite number of orbits of the S^1 -action [6, Proposition 2] and $F_t(S_1(F_t))$ consists of circles centered at 0 except the origin;
- (2) For each point $\mathbf{w} \in S_1(F_t)$, there exist local coordinates (x_1, x_2, x_3, x_4) centered at \mathbf{w} such that F_t is given by

$$(F_t/|F_t|, |F_t|) = (x_1, -x_2^2 + x_3^2 + x_4^2 + c_{t,\mathbf{w}}),$$

where $c_{t,\mathbf{w}} = |F_t(\mathbf{w})|$ for $\mathbf{w} \in S_1(F_t)$ and $0 < t \ll 1$. In particular, $S_1(F_t)$ consists of indefinite fold singularities;

- (3) $S_2(F_t) = \{\mathbf{o}\}$ or \emptyset .

In [6], we focused on the mixed singularity of $f\bar{g}$, where f and g are 2-variable weighted homogeneous complex polynomials which have no common branches, and showed that there exists a deformation F_t of $f\bar{g}$ such that F_t satisfies the above conditions. It is important to notice that there exist polar weighted homogeneous polynomials which do not admit deformations into smooth maps with only Morse singularities, see [4, Theorem 1], [5, Corollary 1 and Corollary 2].

By the condition (2), the absolute value $|F_t|$ of F_t defines a Morse function on the orbit space $(D_\varepsilon^4 \cap F_t^{-1}(D_\delta^2))/S^1$ of the S^1 -action for $t > 0$ and outside the image of the origin, and the indices of the Morse singularities are always 1. Then the handle decomposition of the orbit space according to this Morse function induces a decomposition of $D_\varepsilon^4 \cap F_t^{-1}(D_\delta^2)$ into a tubular neighborhood of a singular fiber over the origin and a finite number of round 1-handles, that is, we have

$$(0.1) \quad D_\varepsilon^4 \cap F_t^{-1}(D_\delta^2) \cong (D_\varepsilon^4 \cap F_t^{-1}(D_{\delta_t}^2)) \cup_\varphi (\sqcup_{i=1}^\ell (\text{round 1-handle})_i)$$

where ℓ is the number of the singularities of the Morse function on the orbit space outside the origin and $\varphi = (\varphi_1, \dots, \varphi_\ell)$ is the attaching map of ℓ copies of a round 1-handle. Here we may assume that the images of $\varphi_1, \dots, \varphi_\ell$ in $\partial(D_\varepsilon^4 \cap F_t^{-1}(D_{\delta_t}^2))$ are disjoint.

In this paper, we describe the structure of this decomposition more precisely. To state our theorem, we introduce the notion of negative link components. Let P be a polar weighted homogeneous polynomial. Then the link of P at the origin is a Seifert link, denoted by $L(P, \mathbf{o})$. A fiber surface of the Seifert link induces an orientation to each link component canonically. A connected component of $L(P, \mathbf{o})$ is called a *positive component* if the orientation of the link component coincides with that of the S^1 -action, and otherwise it is called a *negative component*. Let $|L^+(P, \mathbf{o})|$ and $|L^-(P, \mathbf{o})|$ denote the number of positive components of $L(P, \mathbf{o})$ and the number of negative components of $L(P, \mathbf{o})$, respectively. Then the decomposition is given as follows:

Theorem 1. *Let P be a 2-variable polar and radial weighted homogeneous polynomial which has an isolated singularity at the origin and let F_t be a deformation of P satisfying the conditions (1), (2) and (3). Then*

- (i) $D_\varepsilon^4 \cap F_t^{-1}(D_{\delta_t}^2)$ is diffeomorphic to the disjoint union of a 4-ball and ℓ copies of $S^1 \times B^3$, where B^3 is a 3-ball, and each φ_i of the attaching map $\varphi = (\varphi_1, \dots, \varphi_\ell)$ maps the two attaching regions of the i -th round 1-handle to both of the boundary of the 4-ball and that of the i -th $S^1 \times B^3$, and these $\ell + 1$ connected components are connected by ℓ round 1-handles attached by the map φ ; and
- (ii) the number ℓ of round 1-handles in the decomposition (0.1) is given as

$$\ell = |L^+(P, \mathbf{o})| - |L^+(F_t, \mathbf{o})| = |L^-(P, \mathbf{o})| - |L^-(F_t, \mathbf{o})|$$

for $0 < t \ll 1$.

As we mentioned, in [6], a deformation of a mixed singularity of type $(f\bar{g}, \mathbf{o})$ is given explicitly. In that case, the number ℓ is determined by the polar degree and the radial degree of P as $\ell = \frac{d_r - d_p}{2pq}$. From the decomposition in (0.1), we can observe that the Milnor fiber of (P, \mathbf{o}) is obtained from the Milnor fiber of (F_t, \mathbf{o}) by removing $2d_p$ disks from two connected components of $D_\varepsilon^4 \cap F_t^{-1}(D_{\delta_t}^2)$ and gluing these boundary circles by d_p annuli for each $i = 1, \dots, \ell$. Moreover, we see that monodromy exchanges these ℓ copies of the union of d_p annuli and $2d_p$ disks cyclically.

This paper is organized as follows. In Section 2 we give the definitions of fold singularities and round handles and introduce deformations of polar weighted homogeneous polynomials. In Section 3 we prove Theorem 1. In Section 4 we make a few comments on the monodromy of the Milnor fibration of F_t and a specific deformation of $f\bar{g}$.

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2. PRELIMINARIES

2.1. Fold singularities. Let X be a n -dimensional manifold and W be a 2-dimensional manifold. We denote $C^\infty(X, W)$ the set of smooth maps from X to W . It is known that the subset of smooth maps from X to W whose singularities are only definite fold singularities, indefinite fold singularities or cusps is open and dense in $C^\infty(X, W)$ topologized with the C^∞ -topology [8]. Here a *fold singularity* is the singularity of the following map

$$(x_1, \dots, x_n) \mapsto (x_1, \sum_{j=2}^n \pm x_j^2),$$

where (x_1, \dots, x_n) are coordinates of a small neighborhood of the singularity. If the coefficients of x_j for $j = 2, \dots, n$ is either all positive or all negative, we say that x is a *definite fold singularity*, otherwise it is an *indefinite fold singularity*.

2.2. Deformations of polar weighted homogeneous polynomials. Let $P : \mathbb{C}^2 \rightarrow \mathbb{C}$ be a polar weighted homogeneous polynomial map which has an isolated singularity at the origin of \mathbb{C}^2 . Then P admits a Milnor fibration, i.e., there exist positive real numbers ε and δ such that the map

$$P : D_\varepsilon^4 \cap P^{-1}(\partial D_\delta^2) \rightarrow \partial D_\delta^2$$

is a locally trivial fibration over ∂D_δ^2 , where $D_\varepsilon^4 = \{\mathbf{z} \in \mathbb{C}^2 \mid \|\mathbf{z}\| \leq \varepsilon\}$ and $D_\delta^2 = \{\eta \in \mathbb{C} \mid |\eta| \leq \delta\}$. We fix such positive real numbers ε and δ . Let F be the fiber surface of a polar weighted homogeneous polynomial P . In [14, 15], the monodromy $h : F \rightarrow F$ of the Milnor fibration of P is given by

$$(z_1, z_2) \mapsto \left(\exp\left(\frac{2p\pi i}{d_p}\right) z_1, \exp\left(\frac{2q\pi i}{d_p}\right) z_2 \right),$$

where (p, q) is the polar weight of P . Note that the link $K_P = \partial D_\varepsilon^4 \cap P^{-1}(0)$ is an invariant set of the S^1 -action. So the link K_P is a Seifert link in the 3-sphere [3].

Let F_t be a deformation of P which satisfies the conditions (1), (2) and (3). Since the fiber surface $F_0^{-1}(c)$ intersects ∂D_ε^4 transversely, $F_t^{-1}(c)$ intersects ∂D_ε^4 transversely for each $c \in D_\delta^2$ and $0 \leq t \ll 1$. By the Ehresmann's fibration theorem [17], the map

$$F_t : D_\varepsilon^4 \cap F_t^{-1}(\partial D_\delta^2) \rightarrow \partial D_\delta^2$$

is a locally trivial fibration for $0 \leq t \ll 1$. The polar weight of F_t coincides with that of F_0 for $0 < t \ll 1$. Thus the monodromy of $F_t : D_\varepsilon^4 \cap F_t^{-1}(\partial D_\delta^2) \rightarrow \partial D_\delta^2$ is given by the same S^1 -action on \mathbb{C}^2 for each $0 \leq t \ll 1$.

Lemma 1. *The Milnor fibration $F_t : D_\varepsilon^4 \cap F_t^{-1}(\partial D_\delta^2) \rightarrow \partial D_\delta^2$ is isomorphic to the fibration $F_0/|F_0| : \partial D_\varepsilon^4 \setminus \text{Int}N(K_{F_0}) \rightarrow S^1$ for $0 \leq t \ll 1$, where $N(K_{F_0}) = \{\mathbf{z} \in \partial D_\varepsilon^4 \mid |F_0(\mathbf{z})| \leq \delta\}$.*

Proof. Since the fiber surface $F_0^{-1}(c)$ is transversal to ∂D_ε^4 , $F_t^{-1}(c)$ is transversal to ∂D_ε^4 for any $c \in D_\delta^2$ and $0 \leq t \ll 1$. Fix such a positive real number t . We set

$$\begin{aligned} \partial\mathcal{E}(\delta, \varepsilon) &:= \{(\mathbf{z}, t') \in \mathbb{C}^2 \times [0, t] \mid |F_{t'}(\mathbf{z})| = \delta, \|\mathbf{z}\| \leq \varepsilon\} \\ \partial^2\mathcal{E}(\delta, \varepsilon) &:= \{(\mathbf{z}, t') \in \mathbb{C}^2 \times [0, t] \mid |F_{t'}(\mathbf{z})| = \delta, \|\mathbf{z}\| = \varepsilon\}. \end{aligned}$$

Then the projection

$$\pi' : (\partial\mathcal{E}(\delta, \varepsilon), \partial^2\mathcal{E}(\delta, \varepsilon)) \rightarrow [0, t], \quad (\mathbf{z}, t') \mapsto t'$$

is a proper submersion. By the Ehresmann's fibration theorem [17], π' is a locally trivial fibration over $[0, t]$. Thus the projection π' induces a family of isomorphisms $\psi_{t'} : \partial E_0(\delta, \varepsilon) \rightarrow \partial E_{t'}(\delta, \varepsilon)$ of fibrations such that the following diagram is commutative for $0 \leq t' \leq t$:

$$\begin{array}{ccc} \partial E_0(\delta, \varepsilon) & \xrightarrow{\psi_{t'}} & \partial E_{t'}(\delta, \varepsilon) \\ \downarrow F_0 & & \downarrow F_{t'} \\ S_\delta^1 & = & S_\delta^1 \end{array},$$

where $\partial E_{t'}(\delta, \varepsilon) = \{\mathbf{z} \in \mathbb{C}^2 \mid |F_{t'}(\mathbf{z})| = \delta, \|\mathbf{z}\| \leq \varepsilon\}$. Thus the two fibrations $F_{t'} : D_\varepsilon^4 \cap F_{t'}^{-1}(\partial D_\delta^2) \rightarrow \partial D_\delta^2$ and $F_0 : D_\varepsilon^4 \cap F_0^{-1}(\partial D_\delta^2) \rightarrow \partial D_\delta^2$ are isomorphic for $0 \leq t' \leq t$. By [15, Theorem 37], the two fibrations

$$F_0 : \partial E_0(\delta, \varepsilon) \rightarrow S_\delta^1, \quad F_0/|F_0| : S_\varepsilon^3 \setminus \text{Int}N(K_{F_0}) \rightarrow S^1$$

are isomorphic for $\varepsilon > 0, \delta > 0$ and $\delta \ll \varepsilon$. Thus $F_t : D_\varepsilon^4 \cap F_t^{-1}(\partial D_\delta^2) \rightarrow \partial D_\delta^2$ is isomorphic to $F_0/|F_0| : \partial D_\varepsilon^4 \setminus \text{Int}N(K_{F_0}) \rightarrow S^1$ for $0 \leq t \ll 1$. \square

Lemma 2. *The orbit space of $D_\varepsilon^4 \cap F_t^{-1}(\partial D_\delta^2)$ of the S^1 -action is homeomorphic to a holed 2-sphere for $0 \leq t \ll 1$.*

Proof. The monodromy of $F_t : D_\varepsilon^4 \cap F_t^{-1}(\partial D_\delta^2) \rightarrow \partial D_\delta^2$ is given by the same S^1 -action on \mathbb{C}^2 for each $0 \leq t \ll 1$. By Lemma 1, $(D_\varepsilon^4 \cap F_t^{-1}(\partial D_\delta^2))/S^1$ is homeomorphic to $(\partial D_\varepsilon^4 \setminus \text{Int}N(K_{F_0}))/S^1$.

Since the orbit space $\partial D_\varepsilon^4/S^1$ is homeomorphic to a 2-sphere and K_{F_0} is an invariant set of the S^1 -action, the orbit space $(D_\varepsilon^4 \cap F_t^{-1}(\partial D_\delta^2))/S^1$ is a holed 2-sphere. \square

2.3. Round handles. Let X and Y be n -dimensional smooth manifolds. According to [1, 11], we say that X is obtained from Y by attaching a round k -handle if

(1) there are disk bundles over S^1 , E_s^k and E_u^{n-k-1} ,

(2) there exists an embedding $\varphi : \partial E_s^k \times_{S^1} E_u^{n-k-1} \rightarrow \partial Y$ such that $X \cong Y \cup_\varphi E_s^k \oplus E_u^{n-k-1}$, where $E_s^k \oplus E_u^{n-k-1}$ is the Whitney sum of E_s^k and E_u^{n-k-1} over S^1 . The bundle $E_s^k \oplus E_u^{n-k-1}$ over S^1 is called an n -dimensional round k -handle and φ is called the attaching map of $E_s^k \oplus E_u^{n-k-1}$. Note that a sufficiently small compact neighborhood of a connected component of the set of fold singularities can be regarded as an n -dimensional round handle. In our case, a sufficiently small compact neighborhood of each connected component of $S_1(F_t)$ is regarded as a 4-dimensional round 1-handle.

3. PROOF OF THEOREM 1

3.1. Round 1-handles determined by $S_1(F_t)$. By the condition (3), the origin \mathbf{o} is an isolated singularity of F_t . There exist positive real numbers ε_t and δ_t such that $\delta_t \ll \varepsilon_t$ and the map

$$F_t : D_{\varepsilon_t}^4 \cap F_t^{-1}(\partial D_{\delta_t}^2) \rightarrow \partial D_{\delta_t}^2$$

is a locally trivial fibration over $\partial D_{\delta_t}^2$, where $D_{\varepsilon_t}^4 = \{\mathbf{z} \in \mathbb{C}^2 \mid \|\mathbf{z}\| \leq \varepsilon_t'\}$ and $D_{\delta_t}^2 = \{\eta \in \mathbb{C} \mid |\eta| \leq \delta_t'\}$ for $0 < \varepsilon_t' \leq \varepsilon_t, 0 < \delta_t' \leq \delta_t$ and $\delta_t' \ll \varepsilon_t'$, see [17]. Thus $F_t^{-1}(c)$ intersects $\partial D_{\varepsilon_t}^4$ transversely for any $c \in D_{\delta_t}^2$ and $0 \leq t \ll 1$. We assume that ε_t and δ_t satisfy the following properties:

$$D_{\varepsilon_t}^4 \cap S_1(F_t) = \emptyset, \quad D_{\delta_t}^2 \cap F_t(S_1(F_t)) = \emptyset.$$

See Figure 1. Put $M_0 = D_\varepsilon^4 \cap F_t^{-1}(D_{\delta_t}^2)$.

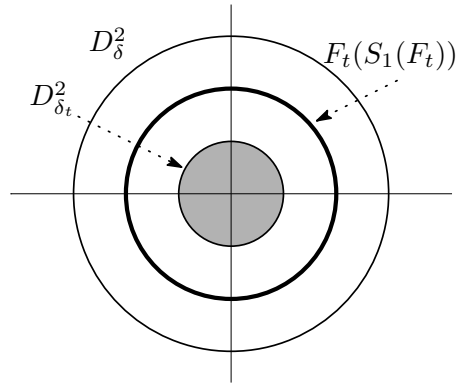


FIGURE 1. $D_{\delta_t}^2$ and $F_t(S_1(F_t))$

Fix t with $0 < t \ll 1$ and let ℓ be the number of singularities of $|F_t|$. Note that $|F_t|$ is a Morse function by the conditions (1) and (2) except the origin. Let C_1, \dots, C_ℓ be the connected components of $S_1(F_t)$, where the number of the connected components of $S_1(F_t)$ is ℓ because of the conditions (1), (2) and (3). We may assume that $|F_t|$ satisfies

$$|F_t(c_1)| \leq |F_t(c_2)| \cdots \leq |F_t(c_\ell)|$$

for $c_i \in C_i$ and $i = 1, \dots, \ell$. Let c'_i be the singularity of $|F_t|$ corresponding to C_i and N'_i be a sufficiently small compact neighborhood of c'_i for $i = 1, \dots, \ell$. Since $|F_t|$ is a Morse function, each $N'_i, i = 1, \dots, \ell$, can be regarded as a 3-dimensional 1-handle $[-1, 1] \times D_i^2$, where D_i^2 is a 2-disk. We set $M'_0 = M_0/S^1$ and $M'_i := M'_{i-1} \cup_{\varphi'_i} N'_i$, where $\varphi'_i : \{\pm 1\} \times D_i^2 \rightarrow \partial M'_{i-1}$ is the attaching map of N'_i for $i = 1, \dots, \ell$. We may assume that $\varphi'_i(\{\pm 1\} \times D_i^2) \subset \partial M'_0$ for $i = 1, \dots, \ell$. Then the orbit space D_ε^4/S^1 is a topological manifold obtained from M'_0 by attaching 3-dimensional 1-handles N'_1, \dots, N'_ℓ . Note that D_ε^4/S^1 is homeomorphic to a 3-ball.

Lemma 3. *Let M_0^* be a connected component of M'_0 . Then $\varphi'_i(\{\pm 1\} \times D_i^2) \not\subset \partial M_0^*$ for $i = 1, \dots, \ell$.*

Proof. Assume that there exist $i \in \{1, \dots, \ell\}$ and a connected component M_0^* of M'_0 such that $\varphi'_i(\{\pm 1\} \times D_i^2)$ is contained in ∂M_0^* . Then the genus of $\partial M'_i$ is greater than 0. After attaching 1-handles, the genus of the boundary of the orbit space does not decrease. Thus the genus of $\partial M'_\ell$ is greater than 0. As Lemma 2, the genus of $\partial M'_\ell$ is equal to 0. This is a contradiction. \square

Let M_i and N_i be 4-dimensional manifolds such that $M_i/S^1 = M'_i$ and $N_i/S^1 = N'_i$ respectively for $i = 1, \dots, \ell$. Then N_i can be regarded as a 4-dimensional round 1-handle and M_i is a manifold obtained from M_{i-1} by attaching N_i for $i = 1, \dots, \ell$. By Lemma 3, N_i connects two connected components of M_0 . Note that M_ℓ is diffeomorphic to $D_\varepsilon^4 \cap F_t^{-1}(D_\delta^2)$.

3.2. The fiber surface of $F_t : D_{\varepsilon_t}^4 \cap F_t^{-1}(\partial D_{\delta_t}^2) \rightarrow \partial D_{\delta_t}^2$. We consider the restricted Milnor fibration $F_t : D_\varepsilon^4 \cap F_t^{-1}(\partial D_{\delta_t}^2) \rightarrow \partial D_{\delta_t}^2$ and connected components of M_0 .

Lemma 4. *Let S_0 be the fiber surface of $F_t : D_\varepsilon^4 \cap F_t^{-1}(\partial D_{\delta_t}^2) \rightarrow \partial D_{\delta_t}^2$. Then S_0 is diffeomorphic to the disjoint union of the fiber surface of $F_t|_{D_{\varepsilon_t}^4 \cap F_t^{-1}(\partial D_{\delta_t}^2)}$ and ℓ copies of an annulus, where ℓ is the number of connected components of $S_1(F_t)$.*

Proof. Let $M_0^0, M_0^1, \dots, M_0^k$ denote the connected components of M_0 such that $\mathbf{o} \in M_0^0$. Then $M_0^0 \cap D_{\varepsilon_t}^4 \neq \emptyset$. The restricted map $F_t : D_{\varepsilon_t}^4 \cap F_t^{-1}(D_{\delta_t}^2) \rightarrow D_{\delta_t}^2$ has a unique singularity at the origin \mathbf{o} of \mathbb{C}^2 . By [10, Lemma 11.3], $D_{\varepsilon_t}^4 \cap F_t^{-1}(\partial D_{\delta_t}^2)$ is homeomorphic to $\partial D_{\varepsilon_t}^4 \setminus \text{Int}N(K_{F_t})$, where $N(K_{F_t}) = \{\mathbf{z} \in \partial D_{\varepsilon_t}^4 \mid |F_t(\mathbf{z})| \leq \delta_t\}$. So any fiber surface of $F_t : D_{\varepsilon_t}^4 \cap F_t^{-1}(\partial D_{\delta_t}^2) \rightarrow \partial D_{\delta_t}^2$ is connected. The boundary of the orbit space $(D_{\varepsilon_t}^4 \cap M_0^0)/S^1$ is homeomorphic to a 2-sphere and

$$M_0^j \cap D_{\varepsilon_t}^4 = \emptyset$$

for $j = 1, \dots, k$.

Let S_0^0 be a fiber surface of $F_t|_{M_0^0 \cap (D_{\varepsilon_t}^4 \cap F_t^{-1}(\partial D_{\delta_t}^2))}$. We divide the surface S_0^0 as follows:

$$S_0^0 = (S_0^0 \cap D_{\varepsilon_t}^4) \cup (S_0^0 \cap (D_\varepsilon^4 \setminus \text{Int}D_{\varepsilon_t}^4)).$$

Since $F_t : (D_\varepsilon^4 \setminus \text{Int}D_{\varepsilon_t}^4) \cap F_t^{-1}(D_{\delta_t}^2) \rightarrow D_{\delta_t}^2$ has no singularities and $F_t^{-1}(c)$ intersects $\partial D_\varepsilon^4 \sqcup \partial D_{\varepsilon_t}^4$ transversely for any $c \in D_{\delta_t}^2$, $F_t^{-1}(c) \cap (D_\varepsilon^4 \setminus \text{Int}D_{\varepsilon_t}^4)$ is diffeomorphic to $F_t^{-1}(0) \cap (D_\varepsilon^4 \setminus \text{Int}D_{\varepsilon_t}^4)$. Note that $F_t^{-1}(0)$ is an invariant set of the S^1 -action and $F_t^{-1}(0)/S^1$ is a 1-dimensional algebraic set. The orbit space $F_t^{-1}(0)/S^1$ is diffeomorphic to $[0, 1]$. Thus the connected component of $F_t^{-1}(c) \cap (D_\varepsilon^4 \setminus \text{Int}D_{\varepsilon_t}^4)$ is diffeomorphic to an annulus. So any connected component of $S_0^0 \cap (D_\varepsilon^4 \setminus \text{Int}D_{\varepsilon_t}^4)$ is an annulus. Since any fiber of F_t intersects $\partial D_{\varepsilon_t}^4$ transversely, $S_0^0 \cap \partial D_{\varepsilon_t}^4$ consists of circles and $S_0^0 \cap (D_\varepsilon^4 \setminus \text{Int}D_{\varepsilon_t}^4)$ is diffeomorphic to $(S_0^0 \cap \partial D_{\varepsilon_t}^4) \times [0, 1]$. So we have

$$\begin{aligned} S_0^0 &= (S_0^0 \cap D_{\varepsilon_t}^4) \cup (S_0^0 \cap (D_\varepsilon^4 \setminus \text{Int}D_{\varepsilon_t}^4)) \\ &\cong (S_0^0 \cap D_{\varepsilon_t}^4) \cup ((S_0^0 \cap \partial D_{\varepsilon_t}^4) \times [0, 1]) \\ &\cong S_0^0 \cap D_{\varepsilon_t}^4. \end{aligned}$$

We consider M_0^j for $j = 1, \dots, k$. The restricted map $F_t : M_0^j \rightarrow D_{\delta_t}^2$ has no singularities. For any $c \in D_{\delta_t}^2 \setminus \{0\}$ and $j = 1, \dots, k$, $F_t^{-1}(c) \cap M_0^j$ is diffeomorphic to $F_t^{-1}(0) \cap M_0^j$. Since $F_t^{-1}(0)$ is an invariant set of the S^1 -action, the orbit space $F_t^{-1}(0)/S^1$ is a 1-dimensional algebraic set. So $F_t^{-1}(0)/S^1$ is diffeomorphic to $[0, 1]$ or S^1 . Assume that $F_t^{-1}(0)/S^1 = S^1$. Then $F_t^{-1}(c)$ is a torus and the orbit space $F_t^{-1}(c)/S^1$ is also a torus. Since the boundary of M_ℓ' is a 2-sphere, this is a contradiction. Let S_0^j denote the fiber surface of $F_t|_{M_0^j}$. Then S_0^j/S^1 is diffeomorphic to $[0, 1]$ and the fiber surface S_0^j is diffeomorphic to an annulus for $j = 1, \dots, k$.

By Lemma 3, each N_i connects two connected components of M_0 . Since M_ℓ is connected, we have $k + 1 - \ell = 1$. Thus the number of connected components of S_0 other than that of $F_t : D_{\varepsilon_t}^4 \cap F_t^{-1}(\partial D_{\delta_t}^2)$ is equal to ℓ . \square

Lemma 5. *The connected component M_0^0 of M_0 is diffeomorphic to a 4-ball and M_0^j is diffeomorphic to $S^1 \times B^3$, where B^3 is a 3-ball, for $j = 1, \dots, \ell$.*

Proof. The two fibrations $F_t : D_{\varepsilon_t}^4 \cap F_t^{-1}(\partial D_{\delta_t}^2) \rightarrow \partial D_{\delta_t}^2$ and $F_t/|F_t| : \partial D_{\varepsilon_t}^4 \setminus (\partial D_{\varepsilon_t}^4 \cap F_t^{-1}(0)) \rightarrow S^1$ are isomorphic for any $0 < \delta' \leq \delta_t$. Thus M_0^0 is diffeomorphic to a 4-ball.

The map $F_t|_{M_0^j}$ has no singularities for $j \neq 0$. Then the fiber surface of $F_t : M_0^j \cap F_t^{-1}(\partial D_{\delta_t}^2) \rightarrow \partial D_{\delta_t}^2$ is diffeomorphic to S_0^j for $0 < \delta' \leq \delta_t$. Since the monodromy of $F_t : M_0^j \cap F_t^{-1}(\partial D_{\delta_t}^2) \rightarrow \partial D_{\delta_t}^2$ is given by the S^1 -action, the orbit space $(M_0^j \cap F_t^{-1}(D_{\delta_t}^2 \setminus \{0\}))/S^1$ is homeomorphic to $(S_0^j/S^1) \times (0, 1]$. By Lemma 4, S_0^j/S^1 is also an annulus. We identify S_0^j/S^1 with $S^1 \times [0, 1]$. Since $S^1 \times (0, 1]$ is diffeomorphic to $D^2 \setminus \{0\}$, $(M_0^j \cap F_t^{-1}(D_{\delta_t}^2 \setminus \{0\}))/S^1$ is homeomorphic to $D^2 \times [0, 1] \setminus (\{0\} \times [0, 1])$, where D^2 is a 2-disk centered at 0. Since $F_t^{-1}(0)$ is an invariant set of the S^1 -action, the orbit space of $F_t^{-1}(0)$ is homeomorphic to $\{0\} \times [0, 1]$. Thus the orbit space of M_0^j is homeomorphic to $D^2 \times [0, 1]$. The manifold M_0^j is diffeomorphic to $S^1 \times B^3$. \square

3.3. The number of connected components of $S_1(F_t)$. To complete the proof of Theorem 1, it is enough to show the equality in Theorem 1 (ii). We set $\tilde{M}_0 = D_\varepsilon^4 \cap F_t^{-1}(\partial D_{\delta_t}^2)$ and $\tilde{M}_i = \tilde{M}_{i-1} \cup_{\varphi_i} \partial N_i$ for $i = 1, \dots, \ell$. Since F_t is polar weighted homogeneous, the monodromy of $F_t|_{\tilde{M}_i}$ is given by the S^1 -action on \mathbb{C}^2 . By the condition (2), a fiber of $|F_t| : N_i' \rightarrow \mathbb{R}$ is as follows:

$$|F_t|^{-1}(|u|) \cap N_i' \cong \begin{cases} \text{two 2-disks} & 0 < c_{t,\mathbf{w}} - |u| \ll 1 \\ \text{an annulus} & 0 < |u| - c_{t,\mathbf{w}} \ll 1 \end{cases}.$$

Since the polar degree of F_t is equal to d_p and N_i is a neighborhood of an orbit of the S^1 -action, a fiber of $F_t : N_i \rightarrow D_\delta^2$ is a d_p -fold cover over a fiber of $|F_t|$. Thus we have

$$F_t^{-1}(u) \cap N_i \cong \begin{cases} (\sqcup_{j=1}^{d_p} D_{1,j}^2) \sqcup (\sqcup_{j=1}^{d_p} D_{2,j}^2) & 0 < c_{t,\mathbf{w}} - |u| \ll 1 \\ \sqcup_{j=1}^{d_p} A_j & 0 < |u| - c_{t,\mathbf{w}} \ll 1 \end{cases},$$

where $D_{k,j}^2$ is a 2-disk and A_j is an annulus for $k = 1, 2$ and $j = 1, \dots, d_p$. By Lemma 3, we may assume that $\sqcup_{j=1}^{d_p} D_{1,j}^2$ is contained in a connected component of M_0 which does not contain $\sqcup_{j=1}^{d_p} D_{2,j}^2$. Let S_i be the fiber surface of $F_t|_{\tilde{M}_i}$ for $i = 0, 1, \dots, \ell$. Set $A' = \sqcup_{j=1}^{d_p} A_j$, $\partial A' = \sqcup_{j=1}^{d_p} \partial A_j$ and $D_k' = \sqcup_{j=1}^{d_p} D_{k,j}^2$ for $k = 1, 2$. Then S_i is the surface obtained from S_{i-1} by replacing $D_1' \sqcup D_2'$ by A' . The Euler characteristic $\chi(S_i)$ of S_i is equal to $\chi(S_{i-1}) - 2d_p$, where d_p is the polar degree of F_t for $i = 1, \dots, \ell$. Thus we have

$$\chi(S_\ell) - \chi(S_0) = -2\ell d_p.$$

Lemma 6. *Let ℓ be the number of connected components of $S_1(F_t)$. Then ℓ is equal to $|L^+(P, \mathbf{o})| - |L^+(F_t, \mathbf{o})|$ and also to $|L^-(P, \mathbf{o})| - |L^-(F_t, \mathbf{o})|$.*

Proof. Since the fibration $F_t|_{\tilde{M}_\ell}$ is isomorphic to $P : D_\varepsilon^4 \cap P^{-1}(\partial D_\delta^2) \rightarrow \partial D_\delta^2$ and $L(P, \mathbf{o})$ is the Seifert link in ∂D_ε^4 , we have

$$\chi(S_\ell) = 1 - \{pq(m+n) - p - q\}(m-n),$$

where $m = |L^+(P, \mathbf{o})|$ and $n = |L^-(P, \mathbf{o})|$, see [3, Theorem 11.1]. By Lemma 4, the fiber surface S_0 of $F_t|_{\tilde{M}_0}$ is diffeomorphic to $S_0^0 \sqcup S_0^1 \sqcup \cdots \sqcup S_0^k$ and S_0^j is an annulus for $j \neq 0$. The Euler characteristic $\chi(S_0)$ of S_0 is equal to $\chi(S_0^0)$. Since the link $L(F_t, \mathbf{o})$ is also the Seifert link in a 3-sphere, $\chi(S_0^0)$ is given by

$$\chi(S_0^0) = 1 - \{pq(m' + n') - p - q\}(m' - n'),$$

where $m' = |L^+(F_t, \mathbf{o})|$ and $n' = |L^-(F_t, \mathbf{o})|$. On the other hand, $\chi(S_\ell)$ is equal to $\chi(S_0) - 2\ell d_p$. The polar degree d_p is equal to $pq(m-n)$ and also to $pq(m' - n')$ [3]. Then we have

$$\begin{aligned} \chi(S_\ell) - \chi(S_0) &= -\{pq(m+n) - p - q\}(m-n) + \{pq(m' + n') - p - q\}(m' - n') \\ &= -pq(m-n)(m+n) + pq(m' - n')(m' + n') \\ &= -d_p\{(m+n) - (m' + n')\} \\ &= -2\ell d_p. \end{aligned}$$

So 2ℓ is equal to $m+n - (m' + n')$. Since $m - m'$ is equal to $n - n'$, ℓ is equal to $m - m'$ and also to $n - n'$. \square

We give an example of Lemma 6 which is considered in [6].

Example 1. Set $f(\mathbf{z}) = z_1^m + z_2^m$ and $g(\mathbf{z}) = z_1 + 2z_2$ where $m \geq 3$. We consider a deformation $F_t = f(\mathbf{z})\overline{g(\mathbf{z})} + t(z_1^m \bar{z}_1 + z_1^{m-1} + \gamma z_2^{m-1})$ of $f(\mathbf{z})\overline{g(\mathbf{z})}$ where $\gamma \in \mathbb{C}$. In [6], we take a coefficient γ of $h(\mathbf{z})$ such that

$$\gamma \neq \frac{-(2\alpha' f(z, 1) - mg(z, 1))(mz\bar{z}^{m-1}r^2 + (m-1)\bar{z}^{m-2} - \alpha' z^m r^2)}{(m-1)(m\bar{z}^{m-1}g(z, 1) - \alpha' f(z, 1))}$$

where $(z^m + 1)\overline{(z + 2)} = \alpha'\overline{(z^m + 1)}(z + 2)$, $\alpha' \in S^1$. Then $S_1(F_t)$ is the set of indefinite fold singularities and the link $S_\varepsilon^3 \cap F_t^{-1}(0)$ is a $(m-1, m-1)$ -torus link, where $S_\varepsilon^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = \varepsilon\}$, $\varepsilon \ll 1$. By Lemma 6, the number of connected components of $S_1(F_t)$ is equal to 1.

Proof of Theorem 1. By Lemma 3 and Lemma 5, $D_\varepsilon^4 \cap F_t^{-1}(D_{\delta_t}^2)$ consists of a 4-ball and ℓ -copies of $S^1 \times B^3$ and the image of the attaching map φ_i of i -th round 1-handle is contained in both of the boundary of a 4-ball and that of $S^1 \times B^3$ for $i = 1, \dots, \ell$. By Lemma 6, the number of connected components of $S_1(F_t)$ is equal to $|L^-(P, \mathbf{o})| - |L^-(F_t, \mathbf{o})|$. \square

4. REMARKS

4.1. Monodromy and characteristic polynomials. Let $h : S \rightarrow S$ be a homeomorphism of a surface S . We define

$$\Delta_*(h) = \frac{\Delta_1(h)}{\Delta_0(h)},$$

where $\Delta_i(h)$ is the characteristic polynomial of the homological map from $H_i(S, \mathbb{Z})$ to itself induced by h for $i = 0, 1$.

Let $h_i : S_i \rightarrow S_i$ be the monodromy of $F_t|_{\tilde{M}_i}$ for $i = 1, \dots, \ell$. Since h_i is given by the S^1 -action on \mathbb{C}^2 , $h_i : S_i \rightarrow S_i$ satisfies the following conditions:

- (I) $h_i(S_{i-1} \setminus (D'_1 \sqcup D'_2)) = S_{i-1} \setminus (D'_1 \sqcup D'_2)$ and $h_i|_{S_{i-1} \setminus (D'_1 \sqcup D'_2)} = h_{i-1}|_{S_{i-1} \setminus (D'_1 \sqcup D'_2)}$,
- (II) $h_i|_{D'_k}$ and $h_i|_{A'}$ are periodic maps which satisfy $D_{k,j}^2 \rightarrow D_{k,j+1}^2$ and $A_j \rightarrow A_{j+1}$

for $i = 1, \dots, \ell, j = 1, \dots, d_p$ and $k = 1, 2$. Here $D_{k,d_p+1}^2 = D_{k,1}^2$ and $A_{d_p+1} = A_1$. We calculate $\Delta_*(h_i)$ from $\Delta_*(h_{i-1})$ by using a round 1-handle N_i .

Lemma 7. *Let S_i be the fiber surface of $F_t|_{\tilde{M}_i}$ and $h_i : S_i \rightarrow S_i$ be the monodromy of $F_t|_{\tilde{M}_i}$ for $i = 1, \dots, \ell$. Then the characteristic polynomial of h_i satisfies*

$$\Delta_*(h_i) = \Delta_*(h_{i-1})(t^{d_p} - 1)^2.$$

Proof. Since S_i is the surface obtained from S_{i-1} by replacing $D'_1 \sqcup D'_2$ by A' and h_i satisfies the above properties, we have

$$\Delta_*(h_i) = \frac{\Delta_*(h_i|_{S_{i-1} \setminus (D'_1 \sqcup D'_2)}) \Delta_*(h_i|_{A'})}{\Delta_*(h_i|_{\partial A'})}.$$

By the condition (II), the monodromy matrices of $H_0(D'_k, \mathbb{Z})$, $H_i(A', \mathbb{Z})$ and $H_i(\partial A', \mathbb{Z})$ are equal to

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

for $k = 1, 2$ and $i = 0, 1$. The characteristic polynomial of the above matrix is equal to $t^{d_p} - 1$. So $\Delta_*(h_i|_{A'})$ and $\Delta_*(h_i|_{\partial A'})$ are equal to 1. As the condition (I), we have

$$\Delta_*(h_i|_{S_{i-1} \setminus (D'_1 \sqcup D'_2)}) \Delta_*(h_{i-1}|_{D'_1}) \Delta_*(h_{i-1}|_{D'_2}) = \Delta_*(h_{i-1}).$$

Thus the characteristic polynomial satisfies

$$\begin{aligned} \Delta_*(h_i) &= \Delta_*(h_i|_{S_{i-1} \setminus (D'_1 \sqcup D'_2)}) \\ &= \Delta_*(h_{i-1})(t^{d_p} - 1)^2. \end{aligned}$$

□

Since the two fibrations $P : D_\varepsilon^4 \cap F_t^{-1}(\partial D_\delta^2) \rightarrow \partial D_\delta^2$ and $F_t : D_\varepsilon^4 \cap F_0^{-1}(\partial D_\delta^2) \rightarrow \partial D_\delta^2$ are isomorphic, we have the following theorem.

Theorem 2. *Let h be the monodromy of $P : D_\varepsilon^4 \cap P^{-1}(\partial D_\delta^2) \rightarrow \partial D_\delta^2$. Then $\Delta_*(h)$ is equal to $\Delta_*(h_0)(t^{d_p} - 1)^{2\ell}$, where ℓ is the number of connected components of $S_1(F_t)$.*

Remark 1. *The algebraic set $P^{-1}(0) \cap \partial D_\varepsilon^4$ is a fibered Seifert link in the 3-sphere. Thus the characteristic polynomial of the monodromy of the Milnor fibration of P at the origin can also be calculated from the splice diagram [3].*

4.2. A specific deformation of $f\bar{g}$. We introduce a deformation of $f\bar{g}$ given in [6], where f and g are 2-variables convenient complex polynomials and $f\bar{g}$ has an isolated singularity at the origin \mathbf{o} . We define the \mathbb{C}^* -action on \mathbb{C}^2 as follows:

$$c \circ (z_1, z_2) := (c^q z_1, c^p z_2), \quad c \in \mathbb{C}^*.$$

Assume that $f(\mathbf{z})$ and $g(\mathbf{z})$ satisfy

$$f(c \circ \mathbf{z}) = c^{pqm} f(\mathbf{z}), \quad g(c \circ \mathbf{z}) = c^{pqn} g(\mathbf{z}), \quad m > n.$$

Then $f(\mathbf{z})$ and $g(\mathbf{z})$ are weighted homogeneous polynomials. Two complex polynomials $f(\mathbf{z})$ and $g(\mathbf{z})$ can be written as

$$f(\mathbf{z}) = \prod_{j=1}^m (z_1^p + \alpha_j z_2^q), \quad g(\mathbf{z}) = \prod_{j=1}^n (z_1^p + \beta_j z_2^q), \quad \gcd(p, q) = 1,$$

where $\alpha_j \neq \alpha_{j'}, \beta_j \neq \beta_{j'} (j \neq j')$ and $\alpha_k \neq \beta_{k'}$ for $1 \leq k \leq m$ and $1 \leq k' \leq n$. The mixed polynomial $f(\mathbf{z})\overline{g(\mathbf{z})}$ is a polar and radial weighted homogeneous polynomial, i.e., $f(\mathbf{z})\overline{g(\mathbf{z})}$ satisfies that $f(s \circ \mathbf{z})\overline{g(s \circ \mathbf{z})} = s^{pq(m-n)}f(\mathbf{z})\overline{g(\mathbf{z})}$ and $f(r \circ \mathbf{z})\overline{g(r \circ \mathbf{z})} = r^{pq(m+n)}f(\mathbf{z})\overline{g(\mathbf{z})}$, where $s \in S^1$ and $r \in \mathbb{R}^*$ [14]. We define a deformation of $f(\mathbf{z})\overline{g(\mathbf{z})}$ as follows:

$$F_t(\mathbf{z}) := f(\mathbf{z})\overline{g(\mathbf{z})} + th(\mathbf{z}),$$

where $0 < t \ll 1$ and

$$h(\mathbf{z}) = \begin{cases} \gamma_1 z_1^{p(m-n)} + \gamma_2 z_2^{q(m-n)} & (g(\mathbf{z}) \neq z_1 + \beta z_2), \\ z_1^m \overline{z_1} + z_1^{m-1} + \gamma z_2^{m-1} & (g(\mathbf{z}) = z_1 + \beta z_2). \end{cases}$$

Then $F_t(\mathbf{z})$ is also a polar weighted homogeneous polynomial with the polar degree $pq(m-n)$. By [6, Theorem 1], there exists $h(\mathbf{z})$ such that $F_t(\mathbf{z})$ satisfies the conditions (1), (2) and (3) for $0 < t \ll 1$. The above deformation F_t of $f\bar{g}$ satisfies that $|L^-(F_t, \mathbf{o})| = 0$. By Lemma 6, the number ℓ of connected components of $S_1(F_t)$ is equal to n . Since the radial degree d_r and the polar degree d_p are equal to $pq(m+n)$ and $pq(m-n)$ respectively, we have the following proposition.

Proposition 1. *Let F_t be the above deformation of $f\bar{g}$. Then the number ℓ of connected components of $S_1(F_t)$ is equal to $\frac{d_r - d_p}{2pq}$, where d_r is the radial degree of $f\bar{g}$ and d_p is the polar degree of $f\bar{g}$.*

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